

IRREDUCIBLE SEMIGROUPS OF POSITIVE OPERATORS ON BANACH LATTICES

NIUSHAN GAO AND VLADIMIR G. TROITSKY

ABSTRACT. The classical Perron-Frobenius theory asserts that an irreducible matrix A has cyclic peripheral spectrum and its spectral radius $r(A)$ is an eigenvalue corresponding to a positive eigenvector. In [Rad99, RR00], this was extended to semigroups of matrices and of compact operators on L_p -spaces. We extend this approach to operators on an arbitrary Banach lattice X . We prove, in particular, that if \mathcal{S} is a commutative irreducible semigroup of positive operators on X containing a compact operator T then there exist positive disjoint vectors x_1, \dots, x_r in X such that every operator in \mathcal{S} acts as a positive scalar multiple of a permutation on x_1, \dots, x_r . Compactness of T may be replaced with the assumption that T is peripherally Riesz, i.e., the peripheral spectrum of T is separated from the rest of the spectrum and the corresponding spectral subspace X_1 is finite dimensional. Applying the results to the semigroup generated an irreducible peripherally Riesz operator T , we show that T is a cyclic permutation on x_1, \dots, x_r , $X_1 = \text{span}\{x_1, \dots, x_r\}$, and if $S = \lim_j b_j T^{n_j}$ for some (b_j) in \mathbb{R}_+ and $n_j \rightarrow \infty$ then $S = c(T|_{X_1})^k \oplus 0$ for some $c \geq 0$ and $0 \leq k < r$. We also extend results of [AAB92, Gro95] about peripheral spectra of irreducible operators.

1. INTRODUCTION

Recall that a square matrix A with non-negative entries is said to be irreducible if no permutation of the basis vectors brings it to a block form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. The classical Perron-Frobenius theory (see, e.g., [AA02, Theorem 8.26]) asserts that for such a matrix, its spectral radius $r(A)$ is non-zero, its peripheral eigenvalues (the ones whose absolute value is $r(A)$) are exactly the m -th roots of unity for some $m \in \mathbb{N}$, and the corresponding eigenspaces are one dimensional. Moreover, the eigenspace for $r(A)$ itself is spanned by a vector whose coordinates are all positive.

There have been numerous extensions and generalizations of Perron-Frobenius Theory. In particular, instead of a positive matrix, one can consider a positive operator on a Banach lattice, or even a family of positive operators. We say that such a family is **ideal irreducible** if it has no common invariant closed non-zero proper ideals; it is **band irreducible** if it has no common invariant proper non-zero bands. In particular,

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a positive operator is ideal irreducible (band irreducible) if it has no invariant proper non-zero closed ideals (respectively, bands). It is easy to see that in case of a single positive matrix, these definitions coincide with irreducibility. We refer the reader to [AA02] for details and terminology on Banach lattices and irreducible operators.

There have been many extensions of Perron-Frobenius Theory to ideal or band irreducible operators on Banach lattices; see, e.g., [NS66, Sch74, dP86, Gro95, AA02, Kit05] etc., and references there. In most of these extensions, it is assumed that the operator is compact, or **power compact** (i.e., some power of it is compact), or, at least, the spectral radius is a pole of the resolvent. There have also been some extensions to semigroups of positive operators. For example, Drnovšek in [Drn01] proved that an ideal irreducible semigroup of compact positive operators must contain a non-quasinilpotent operator.

In [Rad99] and in Sections 5.2 and 8.7 of [RR00], a different approach was used to extend Perron-Frobenius Theory from a single irreducible matrix to an irreducible semigroups of matrices or of compact operators on $L_p(\mu)$ ($1 \leq p < +\infty$). In [Lev09], this approach was applied to order continuous Banach lattices. In the current paper, we extend it to arbitrary Banach lattices. Some of the ideas we use are parallel to those used in [Rad99, RR00], but in many cases we had to develop completely new techniques. Some of our results are new even in the case of $L_p(\mu)$ and in the single operator case. Moreover, we weaken the condition that the semigroup consists entirely of compact operators; we only require that the semigroup contains a compact or even a peripherally Riesz operator. An operator T is said to be **peripherally Riesz** if its **peripheral spectrum** $\sigma_{\text{per}}(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$ consists of isolated eigenvalues of finite multiplicity. This class contains all non-quasinilpotent compact and strictly singular operators.

The paper is structured as follows. Recall that a set of positive operators is $\overline{R^+}$ -**closed** if it is norm closed and it is closed under multiplication by positive scalars. Since taking the $\overline{R^+}$ -closure of a semigroup does not affect its invariant closed ideals, we may assume without loss of generality that our semigroups are $\overline{R^+}$ -closed. In Section 2 we show that the $\overline{R^+}$ -closed semigroup generated by a peripherally Riesz operator either contains the peripheral spectral projection of the operator or a non-zero nilpotent operator of small finite rank. We use this in Section 4 to show that if \mathcal{S} is an $\overline{R^+}$ -closed ideal irreducible semigroup and \mathcal{S} contains a peripherally Riesz or a compact operator, then it contains operators of finite rank. Moreover, it contains “sufficiently many” projections of rank r , where r is the minimal non-zero rank of operators in \mathcal{S} .

In Section 5 we discuss the special case when all such projections have the same range (this is the case when \mathcal{S} is commutative, in particular, when \mathcal{S} is generated by a single operator). We show that, in this case, there are disjoint vectors x_1, \dots, x_r in X_+ such that each operator in the semigroup acts on these vectors as a scalar multiple of a permutation. In particular, $x_0 := x_1 + \dots + x_r$ is a common eigenvector for \mathcal{S} . In Section 6 we show that the dual semigroup $\{S^* : S \in \mathcal{S}\}$ has the same properties under the somewhat stronger condition that \mathcal{S} has a unique projection of rank r (which is still satisfied when \mathcal{S} is commutative). In Section 7, we apply our results to finitely generated semigroups. We completely characterize \mathcal{S} in the case when it is generated by a single peripherally Riesz ideal irreducible operator T ; we show that T acts as a scalar multiple of a cyclic permutation of x_1, \dots, x_r . We improve [AA02, Corollary 9.21] that if S and K are two positive commuting operators such that K is compact and S is ideal irreducible then $r(K) > 0$ and $r(S) > 0$; we show that in this case $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ and $\liminf_n \|S^n x\|^{\frac{1}{n}} > 0$ whenever $x > 0$. In Section 8, we extend the results of the preceding sections to band irreducible semigroups of order continuous operators. In particular, it allows us to improve Grobler's characterization of the peripheral spectrum of a band irreducible power compact operator in [Gro95]. Finally, in Section 9 we investigate the structure of one-sided ideals in an irreducible semigroup.

2. PERIPHERALLY RIESZ OPERATORS

Given a set \mathcal{A} of operators on a Banach space X , we write $\overline{\mathbb{R}^+}\mathcal{A}$ for the smallest $\overline{\mathbb{R}^+}$ -closed semigroup containing \mathcal{A} . In particular, if T is an operator on X , we will write $\overline{\mathbb{R}^+}T$ for the $\overline{\mathbb{R}^+}$ -closed semigroup generated by T . Clearly, $\overline{\mathbb{R}^+}T$ consists of all positive scalar multiples of powers of T and of all the operators of form $\lim_j b_j T^{n_j}$ for some sequence (b_j) in \mathbb{R}_+ and some strictly increasing sequence (n_j) in \mathbb{N} ; these limit operators form the *asymptotic* part of $\overline{\mathbb{R}^+}T$.

Given a semigroup \mathcal{S} in $L(X)$, we will denote by $\min \text{rank } \mathcal{S}$ the minimal rank of non-zero elements of \mathcal{S} ; we write $\min \text{rank } \mathcal{S} = +\infty$ if \mathcal{S} contains no non-zero operators of finite rank. Note that if $T \in \mathcal{S}$ then the ideal generated by T in \mathcal{S} consists of all the operators of form ATB where $A, B \in \mathcal{S} \cup \{I\}$.

A vector $u \in \mathbb{C}^n$ is said *unimodular* if $|u_i| = 1$ for all $i = 1, \dots, n$. Let U_n denote the set of all unimodular vectors in \mathbb{C}^n . Clearly U_n is a group with respect to the coordinate-wise product, with unit $1 = (1, \dots, 1)$. We will need the following standard lemma.

2.1. Lemma. *If $u \in \mathcal{U}_n$ then there exists a strictly increasing sequence (m_j) in \mathbb{N} such that $u^{m_j} \rightarrow 1$.*

Proof. Since \mathcal{U}_n is compact, we can find a subsequence $u^{k_j} \rightarrow v$ for some $v \in \mathcal{U}_n$. Passing to a subsequence, we may assume that $m_j := k_{j+1} - k_j$ is strictly increasing. Then $u^{m_j} = u^{k_{j+1}} u^{-k_j} \rightarrow vv^{-1} = 1$. \square

A square matrix A is **unimodular** if there is a basis in which it is diagonal and the diagonal is a unimodular vector. It follows from Lemma 2.1 that in this case (A^{m_j}) converges to the identity matrix.

2.2. The following observation is based on Lemma 1 of [Rad99] and is critical for our study. Let A be a square matrix with $r(A) = 1$ and $\sigma(A) = \sigma_{\text{per}}(A)$. Using Jordan decomposition of A , we can write $A = U + N$ where U is unimodular, N is nilpotent, and $UN = NU$. By Lemma 2.1, we can find a strictly increasing sequence (m_j) such that $U^{m_j} \rightarrow I$.

Case $N = 0$. In this case, $A = U$, so that $A^{m_j} \rightarrow I$.

Case $N \neq 0$. Let k be such that $N^k \neq 0$ but $N^{k+1} = 0$. Then

$$(1) \quad A^n = (U + N)^n = U^n + \binom{n}{1} U^{n-1} N + \cdots + \binom{n}{k} U^{n-k} N^k.$$

Note that $\lim_n \binom{n}{i} / \binom{n}{k} = 0$ whenever $i < k$. Therefore, if we divide (1) by $\binom{n}{k}$, then every term in the sum except the last one converges to zero as $n \rightarrow \infty$. Denote $r_j = m_j + k$ and $c_j = 1 / \binom{r_j}{k}$, then $\lim_j c_j A^{r_j} = \lim_j U^{r_j-k} N^k = N^k$. We can now summarize as follows.

2.3. Proposition. *Let A be a square matrix with $r(A) = 1$ and $\sigma(A) = \sigma_{\text{per}}(A)$. Then exactly one of the following holds:*

- (i) *A is unimodular and $A^{m_j} \rightarrow I$ for some strictly increasing sequence (m_j) in \mathbb{N} ; or*
- (ii) *There exist a strictly increasing sequence (r_j) in \mathbb{N} and a sequence (c_j) in \mathbb{R}_+ such that $c_j \downarrow 0$ and $c_j A^{r_j}$ converges to a non-zero nilpotent (even square-zero) matrix.*

We will refer to these two cases as “unimodular” and “nilpotent”. In the unimodular case, it is easy to see that every operator in $\overline{\mathbb{R}^+}A$ is a scalar multiple of a unimodular operator. The following proposition describes the asymptotic part of $\overline{\mathbb{R}^+}A$ in the nilpotent case.

2.4. Proposition. *Let A be a square matrix with $r(A) = 1$ and $\sigma(A) = \sigma_{\text{per}}(A)$. Suppose that the nilpotent part N of the Jordan decomposition of A is non-zero. If $B = \lim_j b_j A^{n_j}$ with (n_j) strictly increasing, then B is nilpotent (even square-zero) and $b_j \rightarrow 0$.*

Proof. We will use the notations of 2.2. Recall that the matrix U is unimodular with respect to some basis e_1, \dots, e_n . For $x = \sum_{i=1}^n x_i e_i$, put $\|x\| = \sum_{i=1}^n |x_i|$. Clearly, this is a norm on \mathbb{R}^n and U is an isometry with respect to this norm. It follows from (1) that $\binom{n}{k}^{-1} A^n - U^{n-k} N^k \rightarrow 0$ as $n \rightarrow \infty$. Since U is an isometry and $N^k \neq 0$, the sequence $(\|U^{n-k} N^k\|)_n$, and therefore $(\binom{n}{k}^{-1} \|A^n\|)_n$, is bounded above and bounded away from zero. It follows from $b_j A^{n_j} \rightarrow B$ that the sequence $(b_j \binom{n_j}{k})_j$ is bounded, hence $b_j \rightarrow 0$. It also follows that $b_j \binom{n_j}{k} U^{n_j-k} N^k \rightarrow B$ so that $B^2 = \lim_j \left(b_j \binom{n_j}{k} U^{n_j-k} N^k \right)^2 = 0$ because $UN = NU$ and $N^{2k} = 0$. \square

Let T be an operator on a Banach space X . Recall that T is said to be **Riesz** if its non-zero spectrum consists of isolated eigenvalues with finite-dimensional spectral subspaces. Equivalently, the essential spectral radius $r_{\text{ess}}(T)$ is zero. In particular, compact and strictly singular operators are Riesz. We will be interested in the asymptotic part of $\overline{\mathbb{R}^+}T$, and it is really only determined by the restriction of T to its **peripheral spectral subspace**, i.e., the spectral subspace corresponding to $\sigma_{\text{per}}(T)$. This motivates the following definition: we say that T is **peripherally Riesz** if $r(T) > 0$, $\sigma_{\text{per}}(T)$ is a spectral set (i.e., it is separated from the rest of the spectrum), and the peripheral spectral subspace is finite-dimensional. It is often convenient to assume, in addition, that $r(T) = 1$; this can always be achieved by scaling T . Note that T is peripherally Riesz iff $r_{\text{ess}}(T) < r(T)$; in this case, $\sigma_{\text{per}}(T)$ consists of poles of the resolvent. In particular, every non-quasinilpotent Riesz operator is peripherally Riesz. Applying the results of the first part of this section, we obtain the following two possible structures of the asymptotic part of $\overline{\mathbb{R}^+}T$.

2.5. Proposition. *Suppose that T is peripherally Riesz with $r(T) = 1$. Let $X = X_1 \oplus X_2$, where X_1 and X_2 are the spectral subspaces for $\sigma_{\text{per}}(T)$ and its complement, respectively. Let P be the spectral projection onto X_1 . Then exactly one of the following holds.*

- (i) (“Unimodular” case) $T|_{X_1}$ is unimodular, and each operator in the asymptotic part of $\overline{\mathbb{R}^+}T$ is of form $cU \oplus 0$, where $c \geq 0$ and U is unimodular. Some sequence (T^{m_j}) of powers of T converges to P , P is the only non-zero projection in $\overline{\mathbb{R}^+}T$, and $\overline{\mathbb{R}^+}T$ contains no non-zero quasi-nilpotent operators.

- (ii) (“Nilpotent” case) *The asymptotic part of $\overline{\mathbb{R}^+}T$ is non-trivial. For each operator S with $S = \lim_j b_j T^{n_j}$ with (n_j) strictly increasing, we have $S = B \oplus 0$ where $B \in L(X_1)$ is nilpotent (even square-zero) and $b_j \rightarrow 0$. Also, $\overline{\mathbb{R}^+}T$ contains no projections.*

Proof. Let $T_1 = T|_{X_1}$ and $T_2 = T|_{X_2}$.

(i) Suppose that T_1 is unimodular. Take $S \in \overline{\mathbb{R}^+}T$. As X_1 and X_2 are invariant under S , we can write $S = S_1 \oplus S_2$. Suppose that $S = \lim_j b_j T^{n_j}$ for some (b_j) in \mathbb{R}_+ and some strictly increasing sequence (n_j) in \mathbb{N} . Then $b_j T_1^{n_j} \rightarrow S_1$. It follows that S_1 is a scalar multiple of a unimodular matrix and (b_j) is bounded. It now follows from $r(T_2) < 1$ that $S_2 = \lim_j b_j T_2^{n_j} = 0$. So S is of form $cU \oplus 0$. Furthermore, for every non-zero $S \in \overline{\mathbb{R}^+}T$, the restriction $S|_{X_1}$ is a positive scalar multiple of a unimodular matrix, so that S is not quasinilpotent.

By Proposition 2.3, $T_1^{m_j}$ converges to the identity of X_1 . Since $r(T_2) < 1$ we have $T_2^{m_j} \rightarrow 0$. Therefore, $T^{m_j} \rightarrow P$. Finally, let's show that P is the only non-zero projection in $\overline{\mathbb{R}^+}T$. Suppose $Q \in \overline{\mathbb{R}^+}T$ is a projection. Suppose first that $Q = cT^n$ for some $c > 0$ and $n \in \mathbb{N}$. Then $\frac{1}{c^{m_j}}Q = (\frac{1}{c}Q)^{m_j} = T^{nm_j} \rightarrow P^n = P$; it follows that $c = 1$ and $Q = P$. Suppose now that Q is in the asymptotic part of $\overline{\mathbb{R}^+}T$. Then $Q = Q_1 \oplus 0$ where Q_1 is unimodular and is a projection in $L(X_1)$; hence Q is the identity on X_1 and, therefore, $Q = P$.

(ii) Suppose now that T_1 is not unimodular, hence it has a non-trivial nilpotent part. By Proposition 2.3, there exist sequences (c_j) in \mathbb{R}_+ and (r_j) in \mathbb{N} such that $c_j \rightarrow 0$, (r_j) is strictly increasing, and $(c_j T_1^{r_j})$ converges to a non-zero square-zero operator C on X_1 . It follows from $c_j \rightarrow 0$ and $r(T_2) < 1$ that $c_j T_2^{r_j} \rightarrow 0$. Therefore, $c_j T^{r_j} \rightarrow C \oplus 0$, hence $C \oplus 0$ is in the asymptotic part of $\overline{\mathbb{R}^+}T$.

Suppose that $S = \lim_j b_j T^{n_j}$ for some (b_j) in \mathbb{R}_+ and some strictly increasing (n_j) . Proposition 2.4 applied with $A = T_1$ guarantees that $b_j \rightarrow 0$ and $S|_{X_1}$ is a square-zero operator. Furthermore, $r(T_2) < 1$ implies $S|_{X_2} = \lim_j b_j T_2^{n_j} = 0$. In particular, S cannot be a projection.

It is left to show that if $Q = cT^n$ for some $c > 0$ and $n \in \mathbb{N}$ then Q is not a projection. Suppose it is. It follows from $r(Q) = 1 = r(T^n)$ that $c = 1$, so $Q = T^n$. Hence, the set of all distinct powers of T is finite. It follows from $c_j \rightarrow 0$ that $c_j T^{r_j} \rightarrow 0$, but this contradicts $c_j T^{r_j} \rightarrow C \oplus 0 \neq 0$. \square

2.6. Remark. Suppose that, in addition, $\text{rank } T = \min \text{rank } \overline{\mathbb{R}^+}T < \infty$. Then the nilpotent case in Proposition 2.5 is impossible. Indeed, otherwise $\overline{\mathbb{R}^+}T$ would contain

an operator of the form $C \oplus 0$ where C is a nilpotent operator in $L(X_1)$, hence

$$0 < \text{rank } C \oplus 0 = \text{rank } C < \dim X_1 \leq \text{rank } T$$

since T is an isomorphism on X_1 ; a contradiction. Thus, we have $P \in \overline{\mathcal{R}^+}T$, where P is the spectral projection for X_1 . It follows that $\text{rank } T = \text{rank } P = \dim X_1$, so that $T|_{X_2} = 0$. Hence, $\text{Range } T = X_1$, $\ker T = X_2$, and $\sigma(T)$ consists of $\sigma_{\text{per}}(T)$ and, possibly, zero.

3. $\overline{\mathcal{R}^+}$ -CLOSED SEMIGROUPS ON BANACH SPACES

Throughout this section, we assume that \mathcal{S} is an $\overline{\mathcal{R}^+}$ -closed semigroup of operators on a Banach space X . The following result follows immediately from Proposition 2.5.

3.1. Proposition. *If \mathcal{S} contains a peripherally Riesz operator then \mathcal{S} contains a finite-rank operator.*

In particular, this proposition applies when \mathcal{S} contains a non-quasinilpotent compact or even strictly singular operator.

Can we find not just a finite-rank operator in \mathcal{S} but a finite-rank projection? As in Remark 2.6, if there is a $T \in \mathcal{S}$ such that $\text{rank } T = \min \text{rank } \mathcal{S} < +\infty$ and T is not nilpotent then the spectral projection P for $\sigma_{\text{per}}(T)$ is in \mathcal{S} and $\text{rank } P = \text{rank } T$. The next lemma shows that in this case \mathcal{S} contains “sufficiently many” projections.

3.2. Lemma. *Suppose that $S \in \mathcal{S}$ such that $r := \text{rank } S = \min \text{rank } \mathcal{S} < \infty$ and S is not nilpotent. Then there exist projections P and Q in \mathcal{S} with $\text{rank } P = \text{rank } Q = r$ and $PS = SQ = S$. Moreover, the condition “ S is not nilpotent” may be replaced with “ AS is not nilpotent for some $A \in \mathcal{S}$ ”.*

Proof. Suppose AS is not nilpotent for some $A \in \mathcal{S}$ or $A = I$. Then $r(SA) = r(AS) \neq 0$. Clearly, $\text{rank } AS = \text{rank } SA = r$. It follows from $\text{Range } SA \subseteq \text{Range } S$ and $\text{rank } SA = \text{rank } S$ that $\text{Range } SA = \text{Range } S$. By the preceding remark with $T = SA$, the peripheral spectral projection P of SA is in \mathcal{S} , $\text{rank } P = r$, and $\text{Range } P = \text{Range } SA = \text{Range } S$, hence $PS = S$.

In order to find Q , we pass to the adjoint semigroup $\mathcal{S}^* = \{T^* : T \in \mathcal{S}\}$. Note that \mathcal{S}^* , S^* , and A^* still satisfy all the assumptions of the lemma, so we can find a projection $R \in \mathcal{S}^*$ such that $\text{rank } R = r$ and $RS^* = S^*$. Then $R = Q^*$ for some projection $Q \in \mathcal{S}$ with $\text{rank } Q = r$ and $SQ = S$. \square

3.3. Lemma. *Suppose that \mathcal{S} is an $\overline{\mathcal{R}^+}$ -closed semigroup of matrices such that every non-zero matrix in \mathcal{S} is invertible. Then $\{A \in \mathcal{S} : r(A) = 1\}$ is a closed group.*

Proof. Let $\mathcal{S}_1 := \{A \in \mathcal{S} : r(A) = 1\}$. Take any $A \in \mathcal{S}_1$. Since \mathcal{S} contains no non-zero nilpotent matrices, the nilpotent case in Proposition 2.5 is impossible, hence some sequence of powers A^{m_j} converges to the peripheral spectral projection P of A . In particular, $P \in \mathcal{S}$, hence invertible, so that $P = I$ and $\sigma(A)$ is contained in the unit circle. This yields that A is unimodular. It follows from $A^{m_j-1} = A^{-1}A^{m_j} \rightarrow A^{-1}$ that $A^{-1} \in \mathcal{S}$. Clearly, $\sigma(A^{-1})$ is also contained in the unit circle, so that $A^{-1} \in \mathcal{S}_1$.

Suppose that $0 \neq A \in \mathcal{S}$. Then $\frac{1}{r(A)}A \in \mathcal{S}_1$, and the later matrix is unimodular, so that $|\det A| = r(A)^n$. It follows that for $A \in \mathcal{S}$ we have $A \in \mathcal{S}_1$ iff $|\det A| = 1$. Therefore, \mathcal{S}_1 is closed under multiplication. It also follows that \mathcal{S}_1 is closed. \square

4. IDEAL IRREDUCIBLE SEMIGROUPS CONTAINING FINITE-RANK OPERATORS.

Throughout this section, \mathcal{S} is a semigroup of positive operators on a Banach lattice X . For $x \in X$, the **orbit** of x under \mathcal{S} is defined as $\mathcal{S}x = \{Sx : S \in \mathcal{S}\}$. We will use the following known fact; cf. Lemma 8.7.6 in [RR00] and Proposition 2.1 in [DK09].

4.1. Proposition. *The following are equivalent:*

- (i) \mathcal{S} is ideal irreducible;
- (ii) every non-zero algebraic ideal in \mathcal{S} is ideal irreducible;
- (iii) for any non-zero $x \in X_+$ and $x^* \in X_+^*$ there exists $S \in \mathcal{S}$ such that $\langle x^*, Sx \rangle \neq 0$;
- (iv) $A\mathcal{S}B \neq \{0\}$ for any non-zero $A, B \in L(X)_+$.
- (v) for any $x > 0$, the ideal generated in X by the orbit $\mathcal{S}x$ is dense in X .

Proof. The equivalence of (i) through (iv) is Proposition 2.1 in [DK09]. It is easy to see that (i) \Rightarrow (v) \Rightarrow (iii). \square

4.2. Remark. Suppose that $r := \min \text{rank } \mathcal{S} < +\infty$; let \mathcal{S}_r be the set of all operators of rank r in \mathcal{S} and zero. Then \mathcal{S}_r is an ideal, so that \mathcal{S} is ideal irreducible iff \mathcal{S}_r is ideal irreducible. Also, since the set of all operators of rank r is closed in $L(X)$, if \mathcal{S} is $\overline{\mathbb{R}^+}$ -closed then so is \mathcal{S}_r .

The following fact was proved in [Drn01], see also [AA02, Corollary 10.47].

4.3. Theorem ([Drn01]). *If \mathcal{S} consists of compact quasinilpotent operators then \mathcal{S} is ideal reducible.*

4.4. Theorem. *If \mathcal{S} is ideal irreducible, $\overline{\mathbb{R}^+}$ -closed, and contains a peripherally Riesz operator then $\min \text{rank } \mathcal{S} < +\infty$ and \mathcal{S} contains a projection P with $\text{rank } P = \min \text{rank } \mathcal{S}$.*

Proof. By Proposition 3.1, $r := \min \text{rank } \mathcal{S}$ is finite. By Remark 4.2, \mathcal{S}_r is ideal irreducible and, therefore, Theorem 4.3 guarantees that \mathcal{S}_r contains a non-(quasi)-nilpotent operator. Now apply Lemma 3.2. \square

4.5. Example. The following example shows that, in general, for a peripherally Riesz operator $T \in \mathcal{S}$, the peripheral spectral projection of T need not be in \mathcal{S} . Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and let $\mathcal{S} = \overline{\mathbb{R}^+}\{A, B\}$. Clearly, \mathcal{S} is irreducible and the peripheral spectral projection of A is the identity. We claim that $I \notin \mathcal{S}$. Indeed, \mathcal{S} consists of all positive scalar multiples of products of A and B and their limits. Any product that involves B has rank one or zero; since the set of matrices of rank one or zero is closed, any limit of products involving B is also of rank one or zero. On the other hand, it follows from $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ that if $S = \lim b_j A^{n_j}$ then S is a scalar multiple of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, the only elements of \mathcal{S} of rank two are the scalar multiples of powers of A . Hence $I \notin \mathcal{S}$.

4.6. Corollary. *If \mathcal{S} is ideal irreducible, $\overline{\mathbb{R}^+}$ -closed, and contains a non-zero compact operator then $\min \text{rank } \mathcal{S} < +\infty$ and \mathcal{S} contains a projection P with $\text{rank } P = \min \text{rank } \mathcal{S}$.*

Proof. By Theorem 4.4, it suffices to show that \mathcal{S} contains a non-quasinilpotent compact operator. The set of all compact operators in \mathcal{S} is an ideal, hence is ideal irreducible by Proposition 4.1(ii). Then it contains a non-quasinilpotent operator by Theorem 4.3. \square

Throughout the rest of this section, we assume that \mathcal{S} is an ideal irreducible $\overline{\mathbb{R}^+}$ -closed semigroup with $r := \min \text{rank } \mathcal{S} < +\infty$. We denote by \mathcal{S}_r for the ideal of all operators of rank r in \mathcal{S} or zero; we will write \mathcal{P}_r for the (non-empty) set of all projections of rank r in \mathcal{S} .

4.7. Lemma. *For every non-zero $S \in \mathcal{S}_r$ there exists $A \in \mathcal{S}$ such that AS is not nilpotent.*

Proof. Let $\mathcal{J} = \mathcal{S}S\mathcal{S}$. Then \mathcal{J} consists of operators of finite rank, hence compact. \mathcal{J} is non-zero by Proposition 4.1(iv) and ideal irreducible by Proposition 4.1(ii). Hence, by Theorem 4.3, \mathcal{J} contains a non-quasinilpotent operator. That is, there exist $A_1, A_2 \in \mathcal{S}$ such that $0 \neq r(A_1SA_2) = r(A_2A_1S) = r(AS)$ where $A = A_2A_1$. \square

Combining this lemma with Lemma 3.2, we show that \mathcal{S} contains “sufficiently many” rank r projections (cf. Lemmas 5.2.2 and 8.7.17 in [RR00]).

4.8. Theorem. *For every $S \in \mathcal{S}_r$ there exist $P, Q \in \mathcal{P}_r$ such that $PS = SQ = S$.*

4.9. Corollary. *For every non-zero $x \in X_+$ and $x^* \in X_+^*$ there exist $P, Q \in \mathcal{P}_r$ such that $Qx \neq 0$ and $P^*x^* \neq 0$.*

Proof. Since \mathcal{S}_r is ideal irreducible, by Proposition 4.1(iii) there exists $S \in \mathcal{S}_r$ such that $x^*(Sx) \neq 0$. Now take P and Q as in Theorem 4.8. \square

Now, as we know that \mathcal{S} contains “sufficiently many” positive projections of finite rank, we will need to understand the structure of the range of such a projection. The following observation is based on Proposition 11.5 on p. 214 of [Sch74].

4.10. Structure of a positive projection. Let P be a positive projection on X ; let $Y = \text{Range } P$. It is easy to see that Y is a lattice subspace of X with lattice operations $x \wedge^* y = P(x \wedge y)$ and $x \vee^* y = P(x \vee y)$ for any $x, y \in Y$. We will denote this vector lattice by X_P . Note that this lattice structure is determined by Y , so that if Q is another positive projection on X with $\text{Range } Q = Y$ then it generates the same lattice structure on Y .

Suppose, in addition, that $n := \text{rank } P < \infty$. Being a finite-dimensional Archimedean vector lattice, X_P is lattice isomorphic to \mathbb{R}^n with the standard order, see, e.g., [Sch74, Corollary 1, p. 70]. Thus, we can find positive $*$ -disjoint $x_1, \dots, x_n \in X_P$ that form a basis of X_P . Furthermore, we can find positive $y_1^*, \dots, y_n^* \in X_P^*$ such that $y_i^*(x_j) = \delta_{ij}$. Put $x_i^* = y_i^* \circ P$, then $x_1^*, \dots, x_n^* \in X_+^*$ and $x_i^*(x_j) = \delta_{ij}$. It is easy to see that $P = \sum_{i=1}^n x_i^* \otimes x_i$.

Consider $\mathcal{S}_P = \{PSP|_{X_P} : S \in \mathcal{S}\}$, so that $\mathcal{S}_P \subseteq L_+(X_P)$ (note that P need not be in \mathcal{S}). The following proposition extends Lemmas 5.2.1 and 8.7.16 in [RR00].

4.11. Proposition. *If P is a positive finite-rank projection and $P\mathcal{S}P \subseteq \mathcal{S}$ then \mathcal{S}_P is an irreducible $\overline{\mathbb{R}^+}$ -closed semigroup in $L_+(X_P)$.*

Proof. It follows from $P\mathcal{S}P \subseteq \mathcal{S}$ that \mathcal{S}_P is a semigroup. Let $P = \sum_{i=1}^n x_i^* \otimes x_i$ as before; relative to the basis x_1, \dots, x_n , we can view \mathcal{S}_P as a semigroup of positive $n \times n$ matrices. Since \mathcal{S} is ideal irreducible, by Proposition 4.1(iii), for each i, j there exists $S \in \mathcal{S}$ such that $x_i^*(Sx_j) \neq 0$, i.e., the (ij) -th entry of the matrix of $PSP|_{X_P}$ is non-zero. Hence, \mathcal{S}_P is irreducible by Proposition 4.1(iii).

To show that \mathcal{S}_P is closed, suppose that $PS_nP|_{X_P} \rightarrow A$ for some sequence (S_n) in \mathcal{S} and some $A \in L(X_P)$. Put $S = PAP \in L(X)$. Then $PS_nP \rightarrow S$, so that $S \in \mathcal{S}$ because \mathcal{S} is closed. Now $A = PSP|_{X_P}$ yields $A \in \mathcal{S}_P$. \square

Of course, the assumption that $P\mathcal{S}P \subseteq \mathcal{S}$ is satisfied when $P \in \mathcal{S}$. If, in addition, $\text{rank } P = r$, we get the following much stronger result. We write $\mathcal{G}_P := \{PSP|_{X_P} : S \in \mathcal{S} \text{ and } r(PSP) = 1\}$.

4.12. Proposition. *Suppose that $P \in \mathcal{P}_r$. Then every non-zero element of \mathcal{S}_P is invertible and, after appropriately scaling the basis vectors of X_P , \mathcal{G}_P is a transitive¹ group of permutation matrices.*

Proof. By Proposition 4.11, \mathcal{S}_P is irreducible and $\overline{\mathbb{R}^+}$ -closed. Since $r = \min \text{rank } \mathcal{S}$, every non-zero element of \mathcal{S}_P is invertible. It follows from Lemma 3.3 that \mathcal{G}_P is a group. In particular, each matrix in \mathcal{G}_P has a positive inverse. It is known that a positive matrix A in $M_r(\mathbb{R})$ has a positive inverse iff it is a weighted permutation matrix with positive weights, i.e., there exist positive weights w_1, \dots, w_r and a permutation σ of $\{1, \dots, r\}$ such that $Ax_i = w_i x_{\sigma(i)}$ for each $i = 1, \dots, r$.

It is left to show that, after scaling x_i 's, we may assume that all the weights are equal to one (for all $S \in \mathcal{G}_P$). We essentially follow the proof of Lemma 5.1.11 in [RR00]. Since \mathcal{S}_P is an irreducible semigroup of matrices, for each $i, j \leq r$ there exists $A \in \mathcal{S}_P$ such that Ax_i is a scalar multiple of x_j . Put $A_1 = I$. For each $2 = 1, \dots, r$ fix $A_i \in \mathcal{G}_P$ such that $A_i x_1 = \mu_i x_i$ for some $\mu_i > 0$. Replacing x_i with $\mu_i x_i$ for $i = 2, \dots, r$, we have $A_i x_1 = x_i$. It suffices to show that with respect to these modified x_i 's, all the matrices in \mathcal{G}_P are permutation matrices. Let $B \in \mathcal{G}_P$. We know that B is a weighted permutation matrix. Take any i and j such that $\lambda := b_{ij}$ is non-zero. Put $C = A_i^{-1} B A_j$. Then $C \in \mathcal{G}_P$ and $Cx_1 = \lambda x_1$, so that $\lambda = c_{11} \leq r(C) = 1$. Similarly, λ^{-1} is the $(1, 1)$'s entry of C^{-1} , hence $\lambda^{-1} \leq 1$ as well, so that $\lambda = 1$.

Finally, transitivity of \mathcal{G}_P follows from the irreducibility of \mathcal{S}_P . □

4.13. Remark. It follows that the vector $x_0 = x_1 + \dots + x_r$ is invariant under \mathcal{G}_P . Furthermore, for each $S \in \mathcal{S}$, if $PSP \neq 0$ then the minimality of rank implies that PSP is an isomorphism on X_P , so that $r(PSP) \neq 0$ and, therefore, a scalar multiple of PSP is in \mathcal{G}_P . It follows that x_0 is a common eigenvector for \mathcal{S}_P with $PSPx_0 = r(PSP)x_0$.

5. SEMIGROUPS WITH ALL THE RANK r PROJECTIONS HAVING THE SAME RANGE

As in the previous section, \mathcal{S} will stand for an $\overline{\mathbb{R}^+}$ -closed ideal irreducible semigroup of positive operators on a Banach lattice, with $r := \min \text{rank } \mathcal{S} < \infty$. We will write

¹Transitive in the sense that for each i and j there exists $A \in \mathcal{G}_P$ such that $Ax_i = x_j$.

\mathcal{S}_r for the (ideal irreducible) ideal of all operators of rank r in \mathcal{S} and zero, and \mathcal{P}_r for the set of all projections of rank r in \mathcal{S} (which is non-empty by, e.g., Corollary 4.6).

Let $P \in \mathcal{P}_r$ and x_0 be as in Remark 4.13. For x_0 to be a common eigenvector of the entire semigroup \mathcal{S} it would suffice that $\text{Range } P$ is invariant under S and that $PSP \neq 0$ for every non-zero $S \in \mathcal{S}$. We will see that, surprisingly, the former implies the latter. The following proposition extends Lemmas 5.2.4 and 8.7.18 in [RR00].

5.1. Proposition. *The following are equivalent.*

- (i) *All projections in \mathcal{P}_r have the same range;*
- (ii) *All non-zero operators in \mathcal{S}_r have the same range;*
- (iii) *$S(\text{Range } P) = \text{Range } P$ for all non-zero $S \in \mathcal{S}$ and $P \in \mathcal{P}_r$;*
- (iv) *The range of some $P \in \mathcal{P}_r$ is \mathcal{S} -invariant;*

Proof. (i) \Rightarrow (ii) follows from Theorem 4.8.

(ii) \Rightarrow (iii) Let $S \in \mathcal{S}$ and $P \in \mathcal{P}_r$. Since \mathcal{S}_r is ideal irreducible, $S\mathcal{S}_r \neq \{0\}$, so that $ST \neq 0$ for some $T \in \mathcal{S}_r$. It follows from $\text{Range } T = \text{Range } P$ that $SP \neq 0$. Since $SP \in \mathcal{S}_r$, we have $\text{Range } SP = \text{Range } P$.

(iii) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (i) Suppose that $\text{Range } P$ is \mathcal{S} -invariant for some $P \in \mathcal{P}_r$. Take any $Q \in \mathcal{P}_r$. We have $Q\mathcal{S}P \neq \{0\}$ by Proposition 4.1(iv), so that $QSP \neq 0$ for some $S \in \mathcal{S}$. By assumption, $SP = PSP$, so that $QPSP \neq 0$, hence $QP \neq 0$. This yields $\text{rank } QP = r$. By assumption, $\text{Range } QP = Q(\text{Range } P) \subseteq \text{Range } P$, but, trivially, $\text{Range } QP \subseteq \text{Range } Q$. Since all the three ranges are r -dimensional, the inclusions are, in fact, equalities, so that $\text{Range } P = \text{Range } QP = \text{Range } Q$. \square

Next, we would like to provide a few examples.

5.2. Example. Suppose that $x, y \in X_+$ and $x^*, y^* \in X_+^*$ such that $x^*(x) = y^*(x) = x^*(y) = y^*(y) = 1$. Let $\mathcal{S}_1 = \{x^* \otimes x, y^* \otimes x, x^* \otimes y, y^* \otimes y\}$. Then \mathcal{S}_1 is a semigroup of projections. Let $\mathcal{S} = \overline{\mathbb{R}^+} \mathcal{S}_1$, the semigroup of all positive scalar multiples of the elements of \mathcal{S}_1 . Clearly, \mathcal{S}_1 is exactly the set of the minimal rank projections in \mathcal{S} , and the ranges of the elements of \mathcal{S} are $\text{span } x$ and $\text{span } y$. In particular, all the ranges are the same iff $x = y$.

5.3. Example. More specifically, take in Example 5.2 $X = \mathbb{R}^2$, $x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $y = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$, and $x^* = y^* = [1, 1]$. Then $\mathcal{P}_r = \mathcal{S}_1 = \{P, Q\}$ where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ are ideal irreducible and have different ranges.

5.4. **Example.** Again in Example 5.2, take $X = \mathbb{R}^2$, $x = y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, and $y^* = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. Then $\mathcal{P}_r = \mathcal{S}_1 = \{P, Q\}$ where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ are both irreducible and have the same range.

5.5. **Example.** Again in Example 5.2, take $X = \mathbb{R}^2$, $x = y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x^* = [1, 0]$, and $y^* = [0, 1]$. Then $\mathcal{P}_r = \mathcal{S}_1 = \{P, Q\}$ where $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Even though neither P nor Q are irreducible, they generate an irreducible semigroup. Note that P and Q have the same range.

For the rest of this section, we assume that all the projections in \mathcal{P}_r have the same range. This condition looks rather strong at the first glance. However, it will follow immediately from Proposition 6.1 that it is satisfied for commutative semigroups, and, in particular, for semigroups generated by a single operator.

We are now going to prove a Banach lattice version of Lemmas 5.2.5 and 8.7.9 as well as Theorems 5.2.6 and 8.7.20 of [RR00]. Denote by Y the common range of the projections in \mathcal{P}_r . For a non-zero $S \in \mathcal{S}$ we denote by S_Y the restriction of S to Y ; we write $\mathcal{S}_Y = \{S_Y : 0 \neq S \in \mathcal{S}\}$ and $\mathcal{G} := \{S_Y : S \in \mathcal{S}, r(S_Y) = 1\}$. Note that $\mathcal{S}_Y = \mathcal{S}_P$ and $\mathcal{G} = \mathcal{G}_P$ for every $P \in \mathcal{P}_r$, cf. 4.10 and Proposition 4.12. In particular, \mathcal{G} is a transitive group of permutation matrices in the appropriate positive basis x_1, \dots, x_r of Y . The following lemma follows immediately from Proposition 5.1(iii).

5.6. **Lemma.** *For each non-zero $S \in \mathcal{S}$, the restriction S_Y is an isomorphism of Y . In particular, $r(S_Y) > 0$ and $\frac{1}{r(S_Y)} S_Y \in \mathcal{G}$.*

It follows, in particular, that \mathcal{S} contains no zero divisors and no non-zero quasi-nilpotent operators.

5.7. **Theorem.** *There exist disjoint positive vectors x_1, \dots, x_r such that every $S \in \mathcal{S}$ acts as a scalar multiple of a permutation on x_i 's.*

Proof. The statement follows immediately from Lemma 5.6 and Proposition 4.12 except for the disjointness of x_i 's. By 4.10, we know that Y is a lattice subspace of X , and the positive vectors x_1, \dots, x_r form a basis of Y and are disjoint in Y . The latter means that for each $i, j \leq r$ we have $P(x_i \wedge x_j) = 0$ for every $P \in \mathcal{P}_r$. It now follows from Corollary 4.9 that $x_i \perp x_j$ in X . \square

Note that the ideal generated by Y is invariant under \mathcal{S} .

5.8. **Corollary.** *The subspace Y is a non-zero finite-dimensional sublattice of X invariant under \mathcal{S}^2 . The ideal generated by Y is dense in X*

²This can be viewed as a Banach lattice version of results in [RT08].

5.9. Corollary. *All the operators in \mathcal{S} have a unique common eigenvector x_0 . Namely, $Sx_0 = r(S_Y)x_0$ for each $S \in \mathcal{S}$. Furthermore, x_0 is positive and quasi-interior.*

Proof. Let x_1, \dots, x_r be as in the theorem. Put $x_0 = x_1 + \dots + x_r$. Since each $S \in \mathcal{S}$ is just a scalar multiple of a permutation on x_i 's, it follows that x_0 is a common eigenvector for \mathcal{S} . The ideal I_{x_0} generated by x_0 is exactly the ideal generated by Y , hence is dense in X ; it follows that x_0 is quasi-interior. It is left to verify uniqueness (of course, up to scaling). Indeed, suppose that y is also a common eigenvector for \mathcal{S} . Then for each $P \in \mathcal{P}_r$ we have $y \in \text{Range } P = Y$. It follows that y is a linear combination of x_i 's. In particular, viewed as an element of \mathbb{R}^r , it is a common eigenvector of the transitive group of permutations \mathcal{G} , so that it has to be of the form $(\lambda, \dots, \lambda)$; it follows that $y = \lambda x_0$. \square

Note that the semigroup in Example 5.3 has no common eigenvectors.

5.10. Other eigenvalues of \mathcal{S} . Since every element of \mathcal{G} is a permutation matrix with respect to the basis x_1, \dots, x_r of Y , its Jordan form is diagonal and unimodular. It follows that every non-zero $S \in \mathcal{S}$ has at least r eigenvalues of modulus $r(S_Y)$ (counting geometric multiplicities). If we scale S so that $r(S_Y) = 1$ then $(S_Y)^{r!}$ is the identity of Y ; it follows that these eigenvalues satisfy $\lambda^{r!} = 1$.

5.11. Block-matrix structure of \mathcal{S} . Let $X_i = \overline{I_{x_i}}$ for each $i = 1, \dots, r$. Then $X = X_1 \oplus \dots \oplus X_r$ is a decomposition of X into pair-wise disjoint closed ideals, and for every non-zero $S \in \mathcal{S}$ the block-matrix of S with respect to this decomposition has exactly one non-zero block in each row and in each column.

5.12. Proposition. *If $T \in \mathcal{S}$ is peripherally Riesz then $r(T_Y) = r(T)$. Furthermore, if $r(T) = 1$ then the component of T corresponding to $\sigma_{\text{per}}(T)$ is unimodular.*

Proof. Without loss of generality, $r(T) = 1$. By Lemma 5.6, \mathcal{S} has no non-zero nilpotent elements. It follows that the nilpotent case in Proposition 2.5 is impossible, hence the peripheral spectral projection P of T is in \mathcal{S} and there is an increasing sequence (m_j) in \mathbb{N} with $T^{m_j} \rightarrow P$. In particular, $(T_Y)^{m_j} \rightarrow P_Y$. It follows from $r(T) = 1$ that $r(T_Y) \leq 1$. Suppose that $r(T_Y) < 1$. Then $(T_Y)^{m_j} \rightarrow 0$, hence $P_Y = 0$. But this contradicts P_Y being an isomorphism by Lemma 5.6. \square

5.13. Corollary. *If every non-zero operator in \mathcal{S} is peripherally Riesz then spectral radius is multiplicative on \mathcal{S} .*

Proof. Let $S, T \in \mathcal{S}$. By Proposition 5.12, $r(S) = r(S_Y)$, $r(T) = r(T_Y)$, and $r(ST) = r(S_Y T_Y)$. Since S_Y and T_Y are scalar multiples of permutation matrices by Theorem 5.7, it follows that $r(S_Y T_Y) = r(S_Y) r(T_Y)$. \square

For each non-zero $S \in \mathcal{S}$ we have $r(S^*) = r(S) \geq r(S_Y) > 0$ by Lemma 5.6. The following is a refinement of this fact.

5.14. Corollary. *For every non-zero $S \in \mathcal{S}$ and $x^* \in X_+^*$, we have $\liminf_n \|S^{*n} x^*\|^{\frac{1}{n}} \geq r(S_Y)$. In particular, S^* is strictly positive.*

Proof. For each n , we have $(S^{*n} x^*)(x_0) = x^*(S^n x_0) = r(S_Y)^n x^*(x_0)$ by Corollary 5.9. Since x_0 is quasi-interior, we have $x^*(x_0) \neq 0$, so that $r(S_Y)^n \leq \frac{\|x_0\|}{x^*(x_0)} \|S^{*n} x^*\|$. The result is now straightforward. \square

5.15. Remark. Let x_1, \dots, x_r be a disjoint positive basis of Y as before. Suppose that $P \in \mathcal{P}_r$, then, as in 4.10, we have $P = \sum_{i=1}^r x_i^* \otimes x_i$ for some positive functionals x_1^*, \dots, x_r^* . Observe that these functionals are disjoint. Indeed, by Riesz-Kantorovich formula, if $i \neq j$ then

$$(x_i^* \wedge x_j^*)(x_0) = \inf \{x_i^*(u) + x_j^*(v) : u, v \in [0, x_0], u + v = x_0\} \leq x_i^*(x_j) + x_j^*(x_0 - x_j) = 0,$$

hence $(x_i^* \wedge x_j^*)(x_0) = 0$. Since x_0 is quasi-interior, it follows that $x_i^* \wedge x_j^* = 0$.

6. SEMIGROUPS WITH A UNIQUE RANK r PROJECTION

As before, we assume that \mathcal{S} is an ideal irreducible $\overline{\mathbb{R}^+}$ -closed semigroup of positive operators on a Banach lattice X with $r = \min \text{rank } \mathcal{S} < +\infty$.

In the previous section we showed that if all the rank r projections have the same range then \mathcal{S} has some nice properties. In this section, we will show that many of these properties are also enjoyed by the dual semigroup $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$ provided that \mathcal{S} has a *unique* projection of rank r . Even though this is, obviously, a stronger assumption, the following proposition implies that it is still satisfied for commutative semigroups. It is analogous to Lemmas 5.2.7 and 8.7.21 of [RR00].

6.1. Proposition. *The following are equivalent:*

- (i) \mathcal{P}_r consists of a single projection;
- (ii) Every $P \in \mathcal{P}_r$ commutes with \mathcal{S} ;
- (iii) Some $P \in \mathcal{P}_r$ commutes with \mathcal{S} .

Proof. (i) \Rightarrow (ii) Suppose that $\mathcal{P}_r = \{P\}$ and let $0 \neq S \in \mathcal{S}$. It follows from Proposition 5.1(iii) that $PSP \neq 0$. Hence, PS and SP are non-zero elements of \mathcal{S}_r . Applying Theorem 4.8 to PS and SP we get $PS = PSP = SP$.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Suppose $P \in \mathcal{P}_r$ commutes with \mathcal{S} . It follows that $PSP = SP$ for all $S \in \mathcal{S}$, hence by Proposition 5.1(iv), all the projections in \mathcal{P}_r have the same range. Therefore, $P = QP = PQ = Q$ for every $Q \in \mathcal{P}_r$. \square

Recall that by Proposition 4.12 and Remark 4.13, for each $P \in \mathcal{P}_r$, there is a basis x_1, \dots, x_n of $X_P = \text{Range } P$ such that the group \mathcal{G}_P can be viewed as a transitive group of permutations of the vectors x_1, \dots, x_r ; it follows that $x_0 = x_1 + \dots + x_n$ is a common eigenvector of every operator in \mathcal{S} which leaves $\text{Range } P$ invariant. Then we observed in Section 5 that if all the projections in \mathcal{P}_r have the same range, then this range is invariant under all operators in \mathcal{S} and, therefore, x_0 is a common eigenvector for \mathcal{S} .

Throughout the rest of the section, we assume that \mathcal{S} has a unique projection P of rank r . This condition allows us to “dualize” the results of Section 5 for \mathcal{S}^* , even though \mathcal{S}^* may not be ideal irreducible.

Suppose $\mathcal{P}_r = \{P\}$. As in Section 5, we denote $Y = \text{Range } P = X_P$. We can write it as $P = \sum_{i=1}^r x_i^* \otimes x_i$ as in Remark 5.15. It is easy to see that P^* is a projection onto $X_{P^*} := \text{Range } P^* = \text{span}\{x_1^*, \dots, x_r^*\}$ in X^* . For every non-zero $S \in \mathcal{S}$, it follows from Proposition 6.1 that $PSP = SP = PS$, so that $P^*S^*P^* = S^*P^*$, and, therefore, X_{P^*} is invariant under S^* . Note that $r(P^*S^*P^*) = r(PSP) = r(S_Y) \neq 0$ by Lemma 5.6. As in Section 5, if $r(S_Y) = 1$ then $S \in \mathcal{G}$ (since P is unique, we write $\mathcal{G}_P = \mathcal{G}$) and S acts as a permutation matrix on x_1, \dots, x_r . It follows from $x_i^*(x_j) = \delta_{ij}$ that S^* acts as a permutation matrix on x_1^*, \dots, x_r^* (namely, as the transpose of the matrix of S on x_1, \dots, x_r). Moreover, since \mathcal{G} is transitive on x_1, \dots, x_r , the group $\mathcal{G}^* := \{S^* : S \in \mathcal{G}\}$ is transitive on x_1^*, \dots, x_r^* . In particular, we have $S^*x_0^* = x_0^*$, where $x_0^* = x_1^* + \dots + x_r^*$.

6.2. Corollary. *For every non-zero $S \in \mathcal{S}$, the operator $\frac{1}{r(S_Y)}S^*$ acts as a permutation of x_1^*, \dots, x_r^* . In particular, $S^*x_0^* = r(S_Y)x_0^*$ for each non-zero $S \in \mathcal{S}$. The functional x_0^* is strictly positive and is a unique common eigenfunctional for \mathcal{S}^* .*

Proof. Uniqueness is proved exactly as in Corollary 5.9. It is left to prove that x_0^* is strictly positive. Fix $x > 0$. By Proposition 4.1(iii), there exists $S \in \mathcal{S}$ with $x_0^*(Sx) \neq 0$. Since $r(S_Y) \neq 0$ by Lemma 5.6 and $x_0^*(Sx) = (S^*x_0^*)x = r(S_Y)x_0^*(x)$, we have $x_0^*(x) \neq 0$. \square

In view of Corollary 6.2, the following fact is the dual version of Corollary 5.14; the proof is analogous. Corollaries 6.2 and 6.3 extend Lemma 5.2.8 and Corollary 8.7.22 in [RR00].

6.3. Corollary. *For every $x > 0$ and every non-zero $S \in \mathcal{S}$ we have $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq r(S_Y)$. In particular, S is strictly positive.*

This means that not only every non-zero $S \in \mathcal{S}$ is not quasi-nilpotent, but it is not even *locally* quasi-nilpotent.

We would like to point out that Corollaries 6.2 and 6.3 generally fail if instead of assuming that \mathcal{S} has a unique minimal projection we only assume, as in Section 5, that all the rank r projections in \mathcal{S} have the same range. Indeed, the semigroups in Examples 5.4 and 5.5 are irreducible, $\overline{\mathbb{R}^+}$ -closed, have exactly two distinct projections P and Q of rank r each, and they have the same range. Nevertheless it is easy to see that the dual semigroup \mathcal{S}^* in Example 5.4 has no common eigenfunctionals (as P^* and Q^* have no common eigenfunctionals), while the operators P and Q in Example 5.5 are not strictly positive.

Recall that a positive operator T is ***strongly expanding*** if Tx is quasi-interior whenever $x > 0$.

6.4. Corollary. *The projection P is strongly expanding iff $r = 1$.*

Proof. Note that P is strictly positive by Corollary 6.3, and the ideal generated by $\text{Range } P$ is dense in X by Corollary 5.8. If $r = 1$ then $\text{Range } P$ is the span of x_1 , hence x_1 is quasi-interior and Px is a positive scalar multiple of x_1 whenever $x > 0$. On the other hand, if $r > 1$ then $Px_1 = x_1 \perp x_2$, hence Px_1 is not quasi-interior. \square

The following proposition should be compared with Proposition 5.12.

6.5. Proposition. *Let $0 \neq S \in \mathcal{S}$. If $r(S)$ is an eigenvalue of S or S^* then $r(S_Y) = r(S)$, and the eigenspace is a sublattice.*

Proof. Suppose that $Sx = r(S)x$ for some $x \neq 0$. It follows from $r(S_Y) \leq r(S)$ that

$$(2) \quad r(S_Y)|x| \leq r(S)|x| = |Sx| \leq S|x|,$$

so that $S|x| - r(S_Y)|x| \geq 0$. On the other hand, Corollary 6.2 yields $x_0^*(S|x| - r(S_Y)|x|) = 0$. Since x_0^* is strictly positive, we have $S|x| = r(S_Y)|x|$. Combining this with (2), we get $r(S_Y) = r(S)$. It also follows that $|x|$ is also in the eigenspace, so that the eigenspace is a sublattice.

The proof in the case when $r(S)$ is an eigenvalue of S^* is similar in view of the fact that x_0 is quasi-interior and, therefore, acts as a strictly positive functional on X^* . \square

6.6. Example. Fix $n > 2$ and let \mathcal{S} be the semigroup of all positive scalar multiples of all permutation matrices in $M_n(\mathbb{R})$. Then \mathcal{S} is not commutative; nevertheless, the identity matrix is the unique element of \mathcal{P}_r .

Commutative semigroups. All the results of Sections 5 and 6 apply to commutative semigroups. In particular, the group \mathcal{G} is a commutative transitive semigroup of permutation matrices. Every matrix in such a group is a direct sum of cycles of equal lengths; it follows, in particular, that S_Y^r is a multiple of the identity on Y for each $S \in \mathcal{S}$. See [RR00, Lemma 5.2.11]) for a proof and further properties of such groups of matrices.

7. APPLICATIONS TO FINITELY GENERATED SEMIGROUPS

Singly generated semigroups. Suppose that T is a positive ideal irreducible peripherally Riesz operator on a Banach lattice X . We now present a version of Perron-Frobenius Theorem for T , extending Corollaries 5.2.3 and 8.7.24 in [RR00]. In addition, we completely describe $\overline{\mathbb{R}^+}T$ (cf. Proposition 2.5). For simplicity, scaling T if necessary, we assume that $r(T) = 1$. Let $X = X_1 \oplus X_2$ be the spectral decomposition for T where X_1 is the subspace for $\sigma_{\text{per}}(T)$, and $T = T_1 \oplus T_2$ the corresponding decomposition of T . Clearly, $\overline{\mathbb{R}^+}T$ is ideal irreducible. Since it is commutative, all the results of Sections 5 and 6 apply to it. We will see that, surprisingly, the asymptotic part of $\overline{\mathbb{R}^+}T$ is very small: it consists of finitely many operators and their positive scalar multiples.

7.1. Theorem. *Under the preceding assumptions, $\dim X_1 = \min \text{rank } \overline{\mathbb{R}^+}T$, X_1 has a basis of disjoint positive vectors x_1, \dots, x_r such that T_1 is a cyclic permutation of x_1, \dots, x_r , and $\overline{\mathbb{R}^+}T$ consists precisely of all the powers of T , of the operators $T_1^k \oplus 0$ for $k = 0, \dots, r-1$, and of their positive scalar multiples (and zero).*

Proof. By Proposition 5.12, T_1 is unimodular. Hence, we are in the unimodular case of Proposition 2.5. In particular, the peripheral spectral projection P is the only projection in the semigroup. It follows that $r := \min \text{rank } \overline{\mathbb{R}^+}T = \dim X_1$, $\mathcal{P}_r = \{P\}$, and X_1 coincides with Y in the notation of Section 5. This implies by Theorem 5.7 and Corollary 5.8 that X_1 is a sublattice generated by some disjoint sequence x_1, \dots, x_r and T_1 is a permutation of x_i 's. We claim that this permutation is a cycle of full length r . Indeed, otherwise, T_1 has a cycle of length $m < r$, i.e., after re-numbering the basis vectors, T_1 acts as a cycle on x_1, \dots, x_m . But then the closed ideal generated by x_1, \dots, x_m is invariant under T and is proper as it is disjoint with x_{m+1}, \dots, x_r .

It follows that T_1^r is the identity of X_1 , so that the set of the distinct powers of T_1 is, in fact, finite. Suppose that $0 \neq S = \lim_j b_j T_1^{n_j}$ for some (b_j) in \mathbb{R}_+ and some strictly increasing (n_j) in \mathbb{N} . By Proposition 2.5, $S|_{X_2} = 0$ and $S|_{X_1} = \lim_j b_j T_1^{n_j}$. Since the set of the distinct powers of T_1 is finite, it follows that $S|_{X_1}$ is a scalar multiple of a power of T_1 . \square

- 7.2. Remark.** (i) The ideal generated by X_1 is, clearly, invariant under T , hence it is dense in X .
- (ii) X_1 is a non-zero finite-dimensional sublattice invariant under T .
- (iii) We observed in the proof that P is the unique projection in the semigroup; it can, actually, be viewed as $T_1^0 \oplus 0$.

7.3. Remark. Suppose that S is a positive ideal irreducible operator such that S^m is compact for some m . Then S is strictly positive by [AA02, Theorem 9.3], hence $S^m \neq 0$. Applying [AA02, Theorem 9.19] to S^m we conclude that $r(S^m) \neq 0$ and, therefore, $r(S) \neq 0$. It follows that S is peripherally Riesz. Therefore, Theorem 7.1 applies to positive ideal irreducible power compact operators.

7.4. Remark. It has been known (see, e.g., [NS66]) that if T is a positive ideal irreducible peripherally Riesz operator then $r(T) > 0$, $\sigma_{\text{per}}(T) = r(T)G$ where G is the set of all m -th roots of unity for some $m \in \mathbb{N}$, and each point in $\sigma_{\text{per}}(T)$ is a simple pole of the resolvent with one-dimensional eigenspace. This can now be easily deduced from Theorem 7.1.

Semigroups generated by two commuting operators. de Pagter showed in [dP86] that every ideal irreducible positive compact operator on a Banach lattice has strictly positive spectral radius. This was extended in [AAB92, Corollary 4.11] (see also Corollary 9.21 in [AA02]) to a pair of operators as follows: suppose that S and K are two non-zero positive commuting operators such that S is ideal irreducible and K is compact, then $r(S) > 0$ and $r(K) > 0$. Moreover, K is not even locally quasinilpotent at any positive non-zero vector x , i.e., $\liminf_n \|K^n x\|^{\frac{1}{n}} > 0$, see e.g., [AA02, Corollary 9.19]. Using the results of the preceding sections, we can now strengthen this conclusion even further.

7.5. Theorem. *Under the preceding assumptions on S and K , there exists a quasi-interior vector $x_0 \in X_+$, a strictly positive functional x_0^* , and a positive real λ such that $Sx_0 = \lambda x_0$, $S^*x_0^* = \lambda x_0^*$, $Kx_0 = r(K)x_0$, and $K^*x_0^* = r(K)x_0^*$. Furthermore, $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K) > 0$ whenever $x > 0$.*

Proof. Let $\mathcal{S} = \overline{\mathbb{R}^+}\{S, K\}$. Then \mathcal{S} is ideal irreducible and commutative, so that all the results of Sections 5 and 6 apply. In particular, by Corollaries 5.9 and 6.2 there exist a quasi-interior vector $x_0 \in X_+$ and a strictly positive functional x_0^* such that $Sx_0 = r(S_Y)x_0$, $S^*x_0^* = r(S_Y)x_0^*$, $Kx_0 = r(K_Y)x_0$, and $K^*x_0^* = r(K_Y)x_0^*$. Now put $\lambda := r(S_Y)$ and note that $r(K_Y) = r(K)$ by Proposition 5.12. Also, observe that $r(S) \geq \lambda > 0$ and $r(K) > 0$ by Lemma 5.6.

It is left to show the “furthermore” clause. Fix $x > 0$. It follows from Corollary 6.3 that $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ and $\liminf_n \|K^n x\|^{\frac{1}{n}} \geq r(K)$. However, we clearly have $\limsup_n \|K^n x\|^{\frac{1}{n}} \leq r(K)$, so that $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$. \square

- 7.6. Remark.** (i) It is easy to see that $\limsup_n \|T^n x\|^{\frac{1}{n}} \leq r(T)$ for every operator T and every vector x . Therefore, the conclusion $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ in the theorem is sharp.
- (ii) Corollary 5.14 yields $\liminf_n \|S^{*n} x^*\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^{*n} x^*\|^{\frac{1}{n}} = r(K)$ whenever $x^* > 0$.
- (iii) Clearly, the result (and the proof) remains valid if we require that K is ideal irreducible instead of S . Moreover, the result can be extended to any ideal irreducible commutative collection of operators containing a compact or a peripherally Riesz operator. In this case, the result will still be valid for every operator S in the collection (with λ depending on S).

8. BAND IRREDUCIBLE SEMIGROUPS

In this section, we will show that most of the results of the preceding sections remain valid if we replace ideal irreducibility with band irreducibility under the additional assumption that all the operators in \mathcal{S} are order continuous. This additional assumption is justified by the following two facts. For $A \subseteq X$ we write I_A and B_A for the ideal and the band generated by A , respectively. Suppose that S is a positive order continuous operator. If S vanishes on a set $A \subseteq X_+$ then S also vanishes on B_A . Furthermore, if J is an S -invariant ideal then the band B_J is still S -invariant.

For the rest of this section, we will assume that \mathcal{S} is a semigroup of positive order continuous operators on a Banach lattice X . We start with a variant of Proposition 4.1 for band irreducibility. Recall that for $x > 0$ we write $B_{\mathcal{S}x}$ for the band generated by the orbit $\mathcal{S}x$ of x under \mathcal{S} .

8.1. Lemma. *\mathcal{S} is band irreducible iff $B_{\mathcal{S}x} = X$ whenever $x > 0$.*

Proof. Suppose that \mathcal{S} is band irreducible. It is easy to see that $B_{\mathcal{S}x}$ is \mathcal{S} -invariant for every $x > 0$, so it suffices to prove that $\mathcal{S}x \neq \{0\}$. For each $S \in \mathcal{S}$, since S is order continuous, its null ideal $N_S = \{x \in X : S|x| = 0\}$ is a band. Therefore, $\bigcap_{S \in \mathcal{S}} N_S$ is a band. It is easy to see that the intersection is \mathcal{S} -invariant, hence it is zero. It follows that for every $x > 0$ there exists $S \in \mathcal{S}$ such that $Sx > 0$, so that $\mathcal{S}x$, and therefore $B_{\mathcal{S}x}$, is non-zero.

For the converse, suppose that B is a non-zero proper \mathcal{S} -invariant band. For each $0 < x \in B$ we have $B_{\mathcal{S}x} \subseteq B$, hence $B_{\mathcal{S}x} \neq X$. \square

8.2. Proposition. *Suppose that \mathcal{S} is band irreducible. Then*

- (i) *every non-zero algebraic ideal in \mathcal{S} is band irreducible;*
- (ii) *for any $x > 0$ in X and every order continuous $x^* > 0$ in X^* there exists $S \in \mathcal{S}$ such that $\langle x^*, Sx \rangle \neq 0$;*
- (iii) *$U\mathcal{S}V \neq \{0\}$ for any non-zero $U, V \in L(X)_+$ provided that U is order continuous.*

Proof. (i) Let \mathcal{J} be an algebraic ideal in \mathcal{S} . Take any $x > 0$. Then $y \in I_{\mathcal{J}x}$ iff there exist $S_1, \dots, S_n \in \mathcal{J}$ and $\lambda \in \mathbb{R}_+$ such that $|y| \leq \lambda(S_1 + \dots + S_n)x$. In this case, for any $S \in \mathcal{S}$ we have $|Sy| \leq \lambda(SS_1x + \dots + SS_nx)$, so that Sy is in $I_{\mathcal{J}x}$. It follows that $I_{\mathcal{J}x}$ and, therefore, $B_{\mathcal{J}x}$ is \mathcal{S} -invariant.

Observe that $\mathcal{J}x$ and, therefore, $B_{\mathcal{J}x}$, is non-zero. Indeed, suppose that $\mathcal{J}x = \{0\}$ and fix any non-zero $T \in \mathcal{J}$. Then for every $S \in \mathcal{S}$ we have $TS \in \mathcal{J}$ so that $TSx = 0$. It follows that T vanishes on $\mathcal{S}x$ and, therefore, on $B_{\mathcal{S}x}$. But $B_{\mathcal{S}x} = X$ by Lemma 8.1, so that $T = 0$; a contradiction.

Thus, the band $B_{\mathcal{J}x}$ is \mathcal{S} -invariant and non-zero, hence $B_{\mathcal{J}x} = X$. Now Lemma 8.1 yields the required result.

(ii) Suppose not. Then x^* vanishes on $\mathcal{S}x$, hence on $B_{\mathcal{S}x}$, so that $x^* = 0$; a contradiction.

(iii) Suppose not, suppose $U\mathcal{S}V = \{0\}$. Since $V \neq 0$, there exists $x > 0$ with $Vx > 0$. Then U vanishes on $\mathcal{S}Vx$ and, therefore, on $B_{\mathcal{S}Vx}$, so that, by Lemma 8.1, $U = 0$; a contradiction. \square

Next, we use the idea of the proof of Lemma 3 of [Gro86] to extend Theorem 4.3 to the band irreducible case.

8.3. Proposition. *If all the operators in \mathcal{S} are compact and quasi-nilpotent then \mathcal{S} is band reducible.*

Proof. Let F be the closed ideal generated by the union of the ranges of all the operators in \mathcal{S} . We may assume, without loss of generality, that $\dim F > 1$ as, otherwise, F is a band and we are done. Applying Theorem 4.3 to the restriction of \mathcal{S} to F , we find a non-zero closed ideal $J \subsetneq F$ such that J is \mathcal{S} -invariant. It follows that B_J is \mathcal{S} -invariant. It is left to show that B_J is proper. Suppose that $B_J = X$. Then for any $x \in X_+$ we have $x_\alpha \uparrow x$ for some net (x_α) in J_+ . Let $S \in \mathcal{S}$. Since S is order continuous, we have $Sx_\alpha \uparrow Sx$. Since S is compact, after passing to a subnet we know that (Sx_α) converges in norm; hence $Sx_\alpha \rightarrow Sx$ in norm. It follows that $Sx \in J$. Since $x > 0$ was arbitrary, it follows that $F \subseteq J$; a contradiction. \square

8.4. One can now easily verify that the results of the previous sections remain true for band irreducible semigroups of order continuous operators with the following straightforward modifications.

- In Corollary 4.9, one has to assume that x^* is *order continuous*.
- In 4.10, we now only consider order continuous projections. It is easy to see that the functionals x_1^*, \dots, x_n^* defined there are also order continuous.
- Proposition 4.11 extends as long as P is order continuous.
- In Corollary 5.8, we now conclude that Y is a closed \mathcal{S} -invariant sublattice of X and the band generated by Y is all of X .
- In Corollary 5.9, we replace “quasi-interior” with “a weak unit”.
- In 5.11, we replace $\overline{I_{x_i}}$ with B_{x_i} .
- In Corollary 5.14 we need to assume that x^* is σ -order continuous, because in this case we still have $x^*(x_0) > 0$ (recall that x_0 is now a weak unit). In particular, S^* is strictly positive on σ -order continuous functionals.
- In Corollary 6.2, the functional x_0^* is now order continuous because $x_0^* = x_1^* + \dots + x_r^*$ and x_1^*, \dots, x_r^* are order continuous.
- Proposition 6.5 remains valid for S . For S^* we can only say that if there is an σ -order continuous eigenfunctional x^* for $r(S)$ then $r(S_Y) = r(S)$ and $|x^*|$ is also in the eigenspace. Indeed, as in the proof of Proposition 6.5, we get

$$r(S_Y)|x^*| \leq r(S)|x^*| \leq S^*|x^*| \quad \text{and} \quad (S^*|x^*| - r(S_Y)|x^*|)(x_0) = 0.$$

Since x^* is σ -order continuous, so are $|x^*|$ and $S^*|x^*|$ (because S is order continuous and $S^*|x^*| = |x^*| \circ S$). It follows that $S^*|x^*| - r(S_Y)|x^*|$ is a σ -order continuous functional in X_+ vanishing on a weak unit x_0 , hence $S^*|x^*| = r(S_Y)|x^*|$. Therefore, $r(S_Y) = r(S)$ and $|x^*|$ is in the eigenspace.

Next, we consider finitely generated semigroups. The difficulty here is that in order to use our previous results, we need $\overline{\mathbb{R}^+}T$ to consist of order continuous operators. However, we do not know whether this follows from the assumption that T itself is order continuous (cf. the counterexample in Section 3 of [KW05]).

8.5. Lemma. *Let S and T be two commuting non-zero positive σ -order continuous operators. If T is band irreducible then S is strictly positive.*

Proof. Suppose not, suppose $Sx = 0$ for some $x > 0$. Without loss of generality, $\|T\| < 1$, so that $z := \sum_{n=0}^{\infty} T^n x$ exists. Clearly, $Tz \leq z$. It follows that B_z is invariant under T and, therefore, $B_z = X$. On the other hand, we have $Sz = 0$, so that S vanishes on B_z , so $S = 0$; a contradiction. \square

There have been several variants of the Perron-Frobenius Theorem for band irreducible operators, see e.g., [Gro86, Gro87, Gro95, Kit05]. The following variant was proved in Theorems 5.2 and 5.3 in [Gro95].

8.6. Theorem (Grobler). *If T is positive, band irreducible, σ -order continuous, and power compact then $r(T) > 0$, $\sigma_{\text{per}}(T) = r(T)G$ where G is the set of all m -th roots of unity for some $m \in \mathbb{N}$, and each point in $\sigma_{\text{per}}(T)$ is a simple pole of the resolvent with one-dimensional eigenspace.*

We can now easily deduce this result (and more) from our techniques. Namely, we claim that T enjoys the conclusion of Theorem 7.1. In particular, the peripheral spectral subspace of T is spanned by disjoint positive vectors and T acts as a scalar multiple of a cyclic permutation on these vectors. This easily implies the conclusion of Theorem 8.6.

Indeed, suppose that T^m is compact. Then T and, therefore, T^m is strictly positive by Lemma 8.5. By Lemma 9.30 of [AA02], all the operators in $\overline{\mathbb{R}^+}T$ are order continuous. Then the results of Section 5 and 6 apply to $\overline{\mathbb{R}^+}T$ (again, the proofs must be adjusted as in 8.4). In particular, $\overline{\mathbb{R}^+}T$ contains no quasinilpotent operators, so that $r(T) > 0$. For simplicity, we can scale T so that $r(T) = 1$. Now, the proof of Theorem 7.1 remains valid for T .

Next, we extend this result beyond power compact operators.

8.7. Lemma. *Suppose that $T \in L(X)_+$ and some power of T is σ -order-to-norm continuous. Then every operator in the asymptotic part of $\overline{\mathbb{R}^+}T$ is σ -order-to-norm continuous.*

Proof. Suppose that T^m is σ -order-to-norm continuous and $S = \lim_j b_j T^{n_j}$. Suppose that $x_k \downarrow 0$. Fix a positive real ε . Fix j such that $n_j \geq m$ and $\|S - b_j T^{n_j}\| < \varepsilon$. Observe that

$$\|Sx_k\| \leq \|S - b_j T^{n_j}\| \|x_k\| + \|b_j T^{n_j} x_k\|$$

Note that $\|S - b_j T^{n_j}\| \|x_k\| \leq \varepsilon \|x_1\|$. On the other hand, $x_k \downarrow 0$ yields $T^m x_k \rightarrow 0$, so that $b_j T^{n_j} x_k = (b_j T^{n_j-m}) T^m x_k \rightarrow 0$ in norm as $k \rightarrow \infty$. It follows that $Sx_k \rightarrow 0$. \square

8.8. Corollary. *Suppose that $T \in L(X)_+$ is peripherally Riesz, band irreducible, and σ -order continuous. If some power of T is σ -order-to-norm continuous then every operator in $\overline{\mathbb{R}^+}T$ is order continuous.*

Proof. By Lemma 8.7, every operator in $\overline{\mathbb{R}^+}T$ is σ -order continuous. It follows from Proposition 2.5 that $\overline{\mathbb{R}^+}T$ contains a non-zero compact operator; denote it by K . By Lemma 8.5, K is strictly positive. The result now follows from Corollary 9.16 of [AA02]. \square

In particular, if T is peripherally Riesz, band irreducible, and σ -order-to-norm continuous with $r(T) = 1$ then $\overline{\mathbb{R}^+}T$ consists of order continuous operators and, in view of the preceding remarks, all the conclusions of Theorem 7.1 remain valid. The proof is analogous. Note that this fact is a generalization of Theorem 8.6 because a compact positive σ -order continuous operator is automatically σ -order-to-norm continuous.

Theorem 8.6 can be extended from power compact to power strictly singular operators³. Suppose that T is strictly singular. It follows from Corollary 3.4.5 on p. 193 of [MN91] that T is order weakly compact, i.e., it takes order intervals into relatively weakly compact sets. Suppose that, in addition, T is σ -order continuous. It is now easy to see that T is σ -order-to-norm continuous. Indeed, suppose that $x_n \downarrow 0$. Then $Tx_n \downarrow 0$ and, by Eberlein-Šmulian Theorem there exists a subsequence (x_{n_k}) such that Tx_{n_k} converges weakly. Since (Tx_{n_k}) is monotone, it converges in norm by Theorem 3.52 of [AB06]. It follows that $Tx_{n_k} \rightarrow 0$, so that $Tx_n \rightarrow 0$. Now Corollary 8.8 yields the following result.

8.9. Corollary. *Suppose that $T \geq 0$ is σ -order continuous, band irreducible, and power strictly singular, and $r(T) = 1$. Then all the conclusions of Theorem 7.1 are valid for T .*

³Note that if T^m is strictly singular for some m then $r_{\text{ess}}(T)^m = r_{\text{ess}}(T^m) = 0$. Hence, every non-quasinilpotent power strictly singular operator is peripherally Riesz.

Finally, we can also extend Theorem 7.5 as follows (it can also be viewed as an extension of Corollary 9.34 in [AA02]).

8.10. Theorem. *Suppose that S and K are two non-zero commuting positive operators such that K is compact, σ -order continuous and band irreducible. Then there exists a weak unit $x_0 \in X_+$, a strictly positive functional x_0^* , and a positive real λ such that $Sx_0 = \lambda x_0$, $S^*x_0^* = \lambda x_0^*$, $Kx_0 = r(K)x_0$, and $K^*x_0^* = r(K)x_0^*$. Furthermore, $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ whenever $x > 0$.*

Proof. Let $\mathcal{S} = \overline{\mathbb{R}^+}\{S, K\}$. Then \mathcal{S} is commutative and band irreducible. By Lemma 8.5, K is strictly positive. It follows from Lemma 9.30 of [AA02] that all the operators in \mathcal{S} are order continuous. Hence, all the results of Sections 5 and 6 apply with the modifications described in 8.4. The rest of the proof is exactly as in Theorem 7.5 with the only exception that, instead of being quasi-interior, x_0 is now a weak unit. \square

8.11. Remark. Using Corollary 5.14, which remains valid for band-irreducible semigroups as long as x^* is σ -order continuous, we can show, as in Remark 7.6(ii), that $\liminf_n \|S^{*n} x^*\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^{*n} x^*\|^{\frac{1}{n}} = r(K)$ whenever $x^* > 0$ is σ -order continuous.

8.12. Remark. As in Theorem 7.5, the result can be extended to any commutative semigroup of σ -order continuous operators containing a band irreducible operator and a non-zero compact operator. Indeed, by Lemma 8.5, the compact operator is strictly positive, so that all the operators in the semigroup are order continuous by [AA02, Lemma 9.30]. Now we can apply results of Sections 5 and 6.

9. ONE-SIDED IDEALS OF \mathcal{S}

Some of the properties of an irreducible semigroup can be characterized in terms of minimal right ideals of \mathcal{S} . Throughout this section, we assume that \mathcal{S} is an $\overline{\mathbb{R}^+}$ -closed ideal irreducible semigroup of positive operators on a Banach lattice X with $r = \min \text{rank } \mathcal{S} < +\infty$. We write \mathcal{P}_r for the set of all projections of rank r in \mathcal{S} .

9.1. Lemma. *Every non-zero (right or left) ideal in \mathcal{S} contains a projection of rank r .*

Proof. Let \mathcal{J} be a right ideal in \mathcal{S} . Take any $0 \neq T \in \mathcal{J}$. Since \mathcal{S}_r is ideal irreducible by Remark 4.2, $T\mathcal{S}_r \neq \{0\}$ by Proposition 4.1(iv). Replacing T with a non-zero operator in $T\mathcal{S}_r$ we may assume without loss of generality that $\text{rank } T = r$. By

Lemma 4.7, there exists $A \in \mathcal{S}$ with $r(TA) = 1$; replacing T with TA we may assume that $r(T) = 1$. Let P be the spectral projection of T for $\sigma_{\text{per}}(T)$, then $P \in \mathcal{S}$ by Remark 2.6. It follows that $\text{rank } P = \text{rank } T = r$ and, therefore, $PT = T$. Also, by Proposition 4.12, PTP is invertible in the sense that there exists $S \in \mathcal{S}$ such that $(PTP)(PSP) = P$. It follows that $TPSP = P$, so that $P \in \mathcal{J}$. The proof for a left ideal is similar because $r(TA) = r(AT)$ and T commutes with P . \square

9.2. Corollary. *Minimal right ideals in \mathcal{S} are exactly of form $P\mathcal{S}$, where $P \in \mathcal{P}_r$. In this case, $P\mathcal{S}$ is an ideal iff $\text{Range } P$ is \mathcal{S} -invariant.*

Proof. Suppose \mathcal{J} is a minimal right ideal. By Lemma 9.1, there is a projection P in \mathcal{J} with $\text{rank } P = r$. Since $P\mathcal{S}$ is a right ideal, by minimality we have $P\mathcal{S} = \mathcal{J}$.

Conversely, suppose that $P \in \mathcal{P}_r$; show that $P\mathcal{S}$ is a minimal right ideal. Suppose that \mathcal{J} is a non-zero right ideal in \mathcal{S} and $\mathcal{J} \subseteq P\mathcal{S}$. Again, by Lemma 9.1, there exists a projection $Q \in \mathcal{J}$ such that $\text{rank } Q = r$. It follows from $Q \in P\mathcal{S}$ that $\text{Range } Q = \text{Range } P$. Therefore, $QP = P$, so that $P \in \mathcal{J}$. Hence, $P\mathcal{S} = \mathcal{J}$.

If $P\mathcal{S}$ is an ideal then $SP = SP^2 \in SP\mathcal{S} \subseteq P\mathcal{S}$ for every $S \in \mathcal{S}$, so that $S(\text{Range } P) = \text{Range } SP \subseteq \text{Range } P$. Conversely, if $\text{Range } P$ is \mathcal{S} -invariant then for any $S, T \in \mathcal{S}$ we have $\text{Range } TPS \subseteq \text{Range } P$, so that $TPS = PTPS \in P\mathcal{S}$; hence $P\mathcal{S}$ is an ideal. \square

The next fact can be viewed as an extension of Proposition 5.1.

9.3. Proposition. *The following are equivalent.*

- (i) *All projections in \mathcal{P}_r have the same range;*
- (ii) *All minimal right ideal in \mathcal{S} are ideals;*
- (iii) *Some minimal right ideal in \mathcal{S} is an ideal;*
- (iv) *\mathcal{S} has a unique minimal right ideal.*

Proof. (i) \Rightarrow (ii) follows from Proposition 5.1 and Corollary 9.2.

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (iv) Suppose that $P\mathcal{S}$ is an ideal for some $P \in \mathcal{P}_r$. Let $Q \in \mathcal{P}_r$. Since $Q\mathcal{S}P \neq \{0\}$ by Proposition 4.1(iv), we have $QSP \neq 0$ for some $S \in \mathcal{S}$. Note that $QSP\mathcal{S}$ is a right ideal and $QSP\mathcal{S} \subseteq Q\mathcal{S}$, it follows from minimality that $QSP\mathcal{S} = Q\mathcal{S}$. On the other hand, since $P\mathcal{S}$ is an ideal, $QSP\mathcal{S} \subseteq P\mathcal{S}$, so that $Q\mathcal{S} \subseteq P\mathcal{S}$. Again, by minimality, we have $Q\mathcal{S} = P\mathcal{S}$.

(iv) \Rightarrow (i) Let $P, Q \in \mathcal{P}_r$. Then $P = P^2 \in P\mathcal{S} = Q\mathcal{S}$, hence $\text{Range } P \subseteq \text{Range } Q$. Similarly, $\text{Range } Q \subseteq \text{Range } P$. \square

Corollary 9.2 and Proposition 9.3 extend Lemmas 5.2.3, 5.2.4(i,ix-xi), and 8.7.18 in [RR00].

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DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, AB, T6G 2G1. CANADA

E-mail address: niushan@ualberta.ca, troitsky@ualberta.ca